Macroeconomics 1, lecture notes 1

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1 Introduction

The main purpose of this lecture note is to present the standard one-sector optimal growth model, often referred to as the neoclassical growth model, also known as the Ramsey-Cass-Koopmans model, and explain how to solve it numerically. This model serves as an introduction to modern dynamic general equilibrium models with optimizing agents. It can also be extended in many ways to study a range of issues related to capital accumulation, wages, and interest rates.

In order to facilitate the learning of the neoclassical growth model, and also to study the main implications of different aggregate production technologies, we first present, in section 2, the Solow growth model. We also show how to augment the basic Solow model with competitive capital and labor markets. We then present, in section 3, the neoclassical growth model. We discuss both the so called planner's problem and different competive equilibria. In section 4, we discuss the concept of representative household in some detail and show that the existence of a representative household is not incompatible with wealth and income heterogeneity.

2 The Solow growth model

2.1 The general version

Aggregate production (GDP) in period t is denoted by Y_t and determined as $Y_t = F(A_t, K_t, L_t)$, where $K_t \geq 0$ is aggregate capital stock, $L_t \geq 0$ aggregate labor and $A_t \geq 0$ captures exogenous technological progress. We assume that A and L growth at constant rates; $A_{t+1} = (1+g)A_t$ and $L_{t+1} = (1+n)L_t$.

The economy is closed. Household income equals aggregate production and is split between consumption C and savings S. Savings are assumed to be a constant fraction s of income and investment I equals savings. That is, $S_t = I_t = sY_t$ and $C_t = (1-s)Y_t$. The capital stock evolves as follows

$$K_{t+1} = I_t + (1-\delta)K_t = sF(A_t, K_t, L_t) + (1-\delta)K_t \tag{1}$$

Parameter $\delta > 0$ is the capital depreciation rate.

Given the initial state of the economy, say $K_0 > 0$, $A_0 > 0$, and $L_0 > 0$, the equilibrium of this model consists of $\{K_t, C_t, I_t, Y_t\}_{t=0}^{\infty}$ that are consistent with (1), $Y_t = F(A_t, K_t, L_t)$, $C_t = (1-s)Y_t$, $I_t = sY_t$, $A_{t+1} = (1+g)A_t$, and $L_t = (1+n)L_t$. Given a specific production function, the parameter values, and (K_0, A_0, L_0), we can solve for these equilibrium sequences numerically up to any finite number of periods by simulating the above equations forward in time.

2.2 Balanced growth

Let us now make some assumptions regarding the aggregate production function. First of all, we assume that it can be written as $F(K_t, A_tL_t)$. This means that we have "labour-augmenting" productivity growth. We also assume that the production function features constant returns to scale (CRS), is continously differentiable, strictly concave ($F_K(K, AL) > 0$, $F_{KK}(K, AL) < 0$ and similarly for L) and satisfies the Inada conditions ($lim_{K->0}F_K(K, AL) = \infty$, $lim_{K->\infty}F_K(K, AL) = 0$ and similarly for L). A production function satisfying these assumptions is sometimes referred to as "neoclassical" production function.

The CRS assumption and the assumption that technology is labour-augmenting allow us to rewrite the capital accumulation in a different and useful way. Specifically, because of CRS we have $\frac{F(K_t, A_tL_t)}{A_tL_t} = F(k_t, 1) \equiv f(k_t)$ where $k_t \equiv \frac{K_t}{A_tL_t}$ denotes capital per "effective labor". So by dividing both sides of (1) by A_tL_t (and by multiplying the left-hand side with $\frac{A_{t+1}L_{t+1}}{A_{t+1}L_{t+1}}$), we can rewrite it as

$$k_{t+1}(1+g)(1+n) = sf(k_t) + (1-\delta)k_t \tag{2}$$

This equation reveals that k_t converges to a steady state where it is constant over time. To see this, rewrite it as $k_{t+1}(1+g)(1+n) - k_t = sf(k_t) - \delta k_t$ and consider what happens to the right hand side when k goes to zero or infinity.[1]

Importantly, in this steady state, output per capita (or per worker) and capital per labor grow at constant rates and capital-to-output ratio is constant. In short, this version of the Solow growth model shows that "balanced growth" is possible. For instance, permanent per capita growth does not require an ever increasing capital-to-output ratio. It is also interesting to note that balanced growth appears to require that technological progress is labour-augmenting.[2]

2.3 Competitive labour and capital markets

One shortcoming of the above model is that it doesn't quite resemble a market economy. For one thing, there are no prices! Related to this, it is silent on capital and labor income shares. However, it is relatively straightforward to extend the model so as to have competitive labour and capital markets.

To this end, let us introduce a representative household and a representative firm that both take prices as given. We discuss the concept of "representative" household (or firm) in section 4. For now, you may just assume that there is a large number of identical households and identical firms. The representative household (firm) behaves like the individual households (firms) in the aggregate. For instance, the consumption of the representative household equals the aggregate (or average) consumption of all individual households.

The representative firms rents capital and labor from a representative household to produce an output good the price of which is normalized to one. Its (static) problem is:

[1]Equation (2) is often approximated as $k_{t+1} - k_t = sf(k_t) - (\delta + n + g)k_t$.

[2]This issue is a bit tricky, however. In some cases, a production function that is e.g. of the form $F(A_tK_t, L_t)$ can be rewritten so that it takes the form $F(K_t, A_tL_t)$. You will have an exercise that should clarify this.

$$\pi_t = \max_{K_t, L_t} \{F(K_t, A_t L_t) - w_t L_t - r_t K_t\}, \tag{3}$$

where π_t denotes profits, w_t the wage rate, and r_t the interest rate. Firm's first-order conditions read as:

$$F_{K_t} = r_t \tag{4}$$

$$F_{L_t} A = w_t \tag{5}$$

The representative household saves a fraction s of its capital and labour income and consumes the rest. We may assume that the households owns the firm and thus receives profits as ewll. Thus its capital holdings evolve as $K_{t+1} = (1-\delta)K_t + s(w_t L_t + r_t K_t + \pi)$ while its consumption is $c_t = (1-s)(w_t L_t + r_t K_t + \pi)$.

Because of CRS, the firm makes zero profits. To prove this, note that by the definition of CRS $F(xK, xAL) = xF(K, AL)$. Since this equation holds for all $x > 0$, it can be differentiated with respect to x resulting in $F_K(xK, xAL)K + F_L(xK, xAL)AL = F(K, AL)$. This equation must also hold at $x = 1$, so we also have $F_K(K, AL)K + f_L(K, AL)AL = F(A, AL)$. Using the firm's optimality conditions, this can be written as $rK + wL = F(K, AL)$. In other words, the firms pays its entire output to the household in the form of wages and interests. This also implies that in terms of the aggregate capital stock and consumption, this model version behaves exactly as the previous one without firms.

2.4 Alternative production functions

Textbook presentations of the Solow model often consider only a Cobb-Douglas production function of the form $K^\alpha (AL)^{1-\alpha}$. It is easy to show that this function satisties the "neoclassical" assumptions listed above. The Cobb-Douglas production function also results in constant capital and labor income shares.

However, the Solow model can also be used to consider alternative production functions, that do not necessarily satisfy all the above assumptions. One often used alternative is the CES (constant elasticity of substitution) production function; $(\alpha K^\rho + (1-\alpha)(AL)^\rho)^{1/\rho}$.

One could also try to extend the model to include additional factors of production. For instance, one could consider a production function of the form $K^{\alpha}(A_L L + A_R R)^{1-\alpha}$, where R could stand for "robots" and A_R would be an exogenous productivity level for robots. In this specification, robots are assumed to be perfect substitutes for human labor. This version of the model could be used to analyze how the increasing productivity of robots influences e.g. per capita growth and the capital share. Naturally, one would have to specify how aggregate savings are split between investments in robots and (old type) capital.

3 The basic neoclassical growth model

3.1 The set-up

Time is discrete and goes to infinity. We denote periods by $t = 1, 2, ...\infty$. There is a unit measure of infinitely lived households that derive utility from consumption and leisure. A unit measure of households means that we have uncountably many infinitesimal households with total measure (mass) normalized to one. These assumptions mean that the behavior of a single household has no effect on aggregate variables and also that averages and aggregates are the same (e.g. the average household consumption equals aggregate consumption). We abstract from technological progress or population growth. They would be relatively easy to introduce under assumptions that guarantee balanced growth by considering e.g. capital and consumption per effective labor'.

The consumption in period t is denoted by c_t, and leisure is given by $l_t = 1 - n_t$. The periodic utility function is $u(c, 1-n)$, where c is consumption and n is labor. The utility function is assumed to be increasing in consumption and leisure, continuously differentiable, strictly concave and satisfy the Inada conditions. Household overall welfare is

$$\sum_{t=1}^{\infty} \beta^{t-1} u(c_t, 1-n_t), \tag{6}$$

where $0 < \beta < 1$ is the discount factor.

Together with capital, k, labor can be combined to produce output good, y. The production function is given by

$$y_t = f(k_t, n_t) \tag{7}$$

We assume that the production function exhibits constant returns to scale, is continuously differentiable, strictly concave and satisfies the Inada conditions. The output good can be costlessly converted into consumption good and capital. Capital depreciates at rate $0 \leq \delta \leq 1$.

The aggregate consistency condition (or aggregate resource constraint) is

$$c_t + k_{t+1} = f(k_t, n_t) + (1-\delta)k_t \tag{8}$$

3.2 The planner's problem

The social planner decides over consumption-savings decisions and labor supply in order to maximize the utility of the representative household. It takes as given the initial capital stock and the aggregate resource constraint. Its problem can thus be stated as follows:

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t, 1-n_t) \tag{9}$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= f(k_t, n_t) + (1-\delta)k_t & (10) \\ c_t &\geq 0, 0 \leq n_t \leq 1 & (11) \\ k_1 &> 0 \text{ given} & (12) \end{aligned}$$

The first set of constraints is the aggregate resource constraint for every period. With standard utility and production functions, the inequality constraints in (11) are not going to be binding. Therefore, in what follows, we just ignore them.

The planner's problem can then be formulated as the following Lagrangian:

$$L = \sum_{t=1}^{\infty} \left\{ \beta^{t-1} u(c_t, 1-n_t) + \lambda_t [f(k_t, n_t) + (1-\delta)k_t - c_t - k_{t+1}] \right\} \tag{13}$$

Taking the first order-conditions with respect to c_t, n_t and k_{t+1} and eliminating λ_t results in the following first-order conditions:

$$\begin{aligned} u_{l_t} &= u_{c_t} f_{n_t} & (14) \\ u_{c_t} &= \beta u_{c_{t+1}} (1 - \delta + f_{k_{t+1}}) & (15) \end{aligned}$$

where u_{c_t}, for instance, denotes the partial derivative of u with respect to consumption in period t. (We use u_l to denote the partial derivative of u with respect to its second argument which is leisure. Hence, we have $\frac{\partial u(c_t,1-n_t)}{\partial n_t} \equiv -u_{l_t}$.)

The first equation states that the marginal utility of leisure should equal the marginal utility of consumption times the marginal productivity of labor. This is the same thing as to say that the marginal rate of substitution between leisure and consumption should equal the marginal rate of transformation. The latter equation is an intertemporal first-order condition which is often referred to as Euler equation. In this case, the Euler equation characterizes the evolution of consumption over time. It reveals, for instance, that consumption increases (decreases) over time if the return to investment is higher (lower) than the subjective discount rate.

In addition to these first-order conditions, we need a transversality condition. Here it can be written as:

$$\lim_{t->\infty} \lambda_t k_{t+1} = 0. \tag{16}$$

Intuitively, the transversality condition tells us that far in the future, the social value of the capital stock must approach zero. Otherwise accumulating more capital would increase welfare.

Heuristically, we can think of the transversality condition as the equivalent of a Kuhn-Tucker condition that would be related to a terminal condition in a finite-horizon version of the problem. A finite horizon version of this problem reads as:

$$\max_{\{c_t,n_t,k_{t+1}\}_{t=1}^T} \sum_{t=1}^{T} \beta^{t-1} u(c_t, 1-n_t) \tag{17}$$

subject to

$$c_t + k_{t+1} = f(k_t, n_t) + (1-\delta)k_t \tag{18}$$

$$k_1 > 0 \tag{19}$$

$$k_{T+1} \geq 0 \tag{20}$$

The last constraint is the terminal condition. Clearly, without some constraint on k_{T+1}, this problem would not be well defined. The optimality conditions now include the Kuhn-Tucker condition $vk_{T+1} = 0$, where v denotes the Kuchn-Tucker multiplier related to the terminal condition.

3.3 Solving the planner's problem

Solving the above problem means finding the (unique) paths of consumption, labor, and capital that maximize household utility subject to the constraints starting from some given initial capital stock k_1. In some very special cases, we can solve the planner's problem analytically (that is, with just paper and pencil). More generally, however, the problem can only be solved numerically (that is, with a computer).

The most straightforward way of solving this problem is to solve the system of non-linear equations that consist of the planner's first-order conditions and the periodic aggregate resource constraints. The problem becomes finite-dimensional by assuming that the model converges to a steady state (where consumption, labor and the capital stock are all constant) in a finite number of periods, say in T periods. However, it is important to make sure that T is large enough so that the solution is not affected by it. Thus, the (numerical) problem is to find $\{c_t, k_{t+1}, n_t\}_{t=1}^{T}$ that solve the following set of equations:

$$c_t + k_{t+1} = f(k_t, n_t) + (1-\delta)k_t \text{ for } t = 1, 2, ..., T-1 \quad (21)$$

$$c_T + k_T = f(k_T, n_T) + (1-\delta)k_T \quad (22)$$

$$u_{l_t} = u_{c_t} f_{n_t} \text{ for } t = 1, 2, ..., T \quad (23)$$

$$u_{c_t} = \beta u_{c_{t+1}}(1 - \delta + f_{k_{t+1}}) \text{ for} t = 1, 2, ..., T-1 \quad (24)$$

This system of equations can be solved by standard root finding algorithms. (The assumption that the allocation converges to a steady state suffices to make sure that the allocation that solves this set of equations also satisfies the transversality condition.)

Notice that the steady state is fully pinned down by just three equations (the steady state versions of the first-order conditions and the aggregate resource constraint).

3.4 Competitive equilibrium with sequential trading

In a decentralized economy, allocations are assumed to be chosen by profit maximizing firms and households that aim to maximize their welfare. We again assume that there is a representative firm that rents capital and labor from the households. Its (static) problem is:

$$\max_{k_t,n_t}\{f(k_t,n_t)+(1-\delta)k_t-w_t n_t-(1+r_t)k_t\}, \tag{25}$$

where w_t is the wage rate and r_t the interest rate. Firm's first-order conditions read as:

$$f_{k_t} = r_t+\delta \tag{26}$$

$$f_{n_t} = w_t \tag{27}$$

Constant returns to scale imply that the firm makes zero profits.

Households can trade every period over next period capital holdings (which are in principle allowed to be negative). They face the following periodic budget constraint:

$$c_t+k_{t+1}=(1+r_t)k_t+n_t w_t \tag{28}$$

In addition, we must impose an additional constraint to rule out so called Ponzi-games, where the household finances its consumption by borrowing (implying $k_t<0$) increasingly large amounts. This is usually done by imposing a constraint that requires the present discounted value of asymptotic asset holdings to be non-negative.

The household problem is then the following:

$$\max_{\{c_t,n_t,k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty}\beta^{t-1}u(c_t,1-n_t) \tag{29}$$

$$s.t. \tag{30}$$

$$c_t+k_{t+1} = (1+r_t)k_t+n_t w_t \text{ for } t=1,2,... \tag{31}$$

$$k_1 > 0 \text{ given} \tag{32}$$

$$\lim_{t->\infty}\left[k_{t+1}\prod_{s=2}^{t}\frac{1}{1+r_s}\right] \geq 0 \tag{33}$$

where the last constraint is the no-Ponzi-game condition.

In order to solve the household problem, we first write the following Lagrangian:

$$L=\sum_{t=1}^{\infty}\left\{\beta^{t-1}u(c_t,1-n_t)+\lambda_t[(1+r_t)k_t+n_t w_t-c_t-k_{t+1}]\right\}. \tag{34}$$

The first-order constraints are

$$\beta^{t-1}u_{c_t} - \lambda_t = 0 \tag{35}$$

$$-\beta^{t-1}u_{l_t} + \lambda_t w_t = 0 \tag{36}$$

$$\lambda_{t+1}(1+r_{t+1}) - \lambda_t = 0 \tag{37}$$

In addition, we again have the transversality condition:

$$\lim_{t->\infty} \lambda_t k_{t+1} = 0. \tag{38}$$

Notice that the above Lagrangian does not take into account the no-Ponzi-game condition. However, it turns out that the transversality condition also implies that the no-Ponzi-game condition holds. To see this, use (35) and (37) to write

$$\lambda_t = u_{c_1} \prod_{s=2}^{t} \frac{1}{1+r_s}. \tag{39}$$

Substituting this into (38) reveals that the transversality condition implies

$$\lim_{t->\infty} \prod_{s=2}^{t} \frac{1}{1+r_s} k_{t+1} = 0. \tag{40}$$

Hence, the transversality condition implies that the no-Ponzi-game condition holds with equality. As a result, (35)-(37) together with either (38) or (40) determine the (unique) optimal solution to the household problem stated above.

We can now define the competitive equilibrium. Given $k_1 > 0$, a competitive equilibrium is an allocation $\{c_t, n_t, k_{t+1}\}_{t=1}^{\infty}$ and factor prices $\{r_t, w_t\}_{t=1}^{\infty}$ such that given prices and k_1, the allocation solves the household problem in (29)-(33), the aggregate consistency condition (10) is satisfied, and the factor prices are determined according to (26) and (27).

It is straightforward to show that the allocation associated with the competitive equilibrium is the same as the solution to the planner's problem. Hence, in order to find the competitive equilibrium, we may just solve the social planner's problem to find $\{c_t, n_t, k_{t+1}\}_{t=1}^{\infty}$ and compute factor prices $\{r_t, w_t\}_{t=1}^{\infty}$ as marginal products of capital and labor.

3.5 Competitive equilibrium with period 1 trading

It is often convenient to define the competitive equilibrium so that households face a single budget constraint. This can be done by assuming that in the first period households trade over a full set of assets that pay one unit of consumption or investment good in a given future period. So define p_t as the price of one unit of consumption in period t in terms of consumption in period 1. Since output good can be costlessly transformed into consumption and capital, p_t is also the price of one unit of capital in period t in terms of either period 1 consumption or period 1 investment.

In this case, the household faces only one present value budget constraint:

$$\sum_{t=1}^{\infty} p_t[(1+r_t)k_t + n_t w_t - c_t - k_{t+1}] = 0. \tag{41}$$

The competitive equilibrium now includes also prices $\{p_t\}_{t=1}^{\infty}$.

The first-order conditions read as

$$\beta^{t-1}u_{c_t} - \lambda p_t = 0 \tag{42}$$

$$-\beta^{t-1}u_{l_t} + \lambda p_t w_t = 0 \tag{43}$$

$$p_{t+1}(1+r_{t+1}) - p_t = 0 \tag{44}$$

where λ is the Lagrange multiplier on the present value budget constraint.

The allocation is the same as in the competitive equilibrium defined above. We prove this by showing that the above present value budget constraint is, in the end, equivalent to the periodic budget constraints (28) together with the no-Ponzi-game condition holding with equality.

Note first that we can write the present value budget constraint as follows:

$$\sum_{t=1}^{\infty}[p_{t+1}(1+r_{t+1}) - p_t]k_{t+1} + p_1(1+r_1)k_1 + \sum_{t=1}^{\infty} p_t(n_t w_t - c_t) = 0.$$

Using (44), this simplifies to

$$p_1(1+r_1)k_1 + \sum_{t=1}^{\infty} p_t(n_t w_t - c_t) = 0. \tag{45}$$

Consider then the periodic budget constraint (31). Solving for k_2 from the period 2 budget constraint, and substituting it to the period 1 budget constraint results in

$$c_1 + \frac{c_2}{(1+r_2)} + \frac{k_3}{(1+r_2)} = (1+r_1)k_1 + w_1 n_1 + \frac{w_2 n_2}{(1+r_2)}.$$

Solving for k_3 from the period 3 budget constraint and substituting it to this budget constraint in turn gives us.

$$c_1 + \frac{c_2}{(1+r_2)} + \frac{c_3}{(1+r_2)(1+r_3)} + \frac{k_4}{(1+r_2)(1+r_3)} = (1+r_1)k_1 + w_1 n_1 + \frac{w_2 n_2}{(1+r_2)} + \frac{w_3 n_3}{(1+r_2)(1+r_3)}$$

This generalizes to

$$\sum_{t=1}^{T} \left(\prod_{s=2}^{t} (1+r_s)^{-1} \left[c_t - w_t n_t \right] \right) + \prod_{s=2}^{T} (1+r_s)^{-1} k_{T+1} - (1+r_1)k_1 = 0$$

Taking the limit and imposing $\lim_{T->\infty} \prod_{s=2}^{T} \frac{1}{1+r_s} k_{T+1} = 0$ now allows us to write the periodic budget constraints as the following present value budget constraint:

$$\sum_{t=1}^{\infty} \left(\prod_{s=2}^{t} (1+r_s)^{-1} \left[c_t - w_t n_t \right] \right) - (1+r_1)k_1 = 0.$$

Solving for p_t using (44) reveals that this is the same budget constraint as (45).

4 Wealth and income heterogeneity in the neoclassical growth model

This neoclassical growth model features a representative household. Having a representative household means that to the extent that we are only interested in aggregate variables (e.g. aggregate labor supply and capital accumulation), it is sufficient to consider the problem of a single household (the representative household). The simplest case is where all households are identical in terms of their preferences and endowments. Then all households are representative households. However, an important property of this framework is that a representative household often exists even if households are different.

In this section, we illustrate this by considering a version of the neoclassical growth model with income and wealth heterogeneity. For simplicity, we consider here a special case with exogenous labor supply and log utility. However, the result that the model allows for a representative household extends to the case with endogenous labor supply and more general utility functions, see e.g. Krusell and Rios-Rull (1999, *American Economic Review* 89(5)).

4.1 The set-up

There are I types of households indexed by $i = 1, 2, ..., I$. Let q^i denote the measure of households of type i. We normalize so that $\sum q^i = 1$. Households are different with respect to their capital holdings and their labor productivity. The problem of a household of type i is

$$\max_{\{c_t^i, a_{t+1}^i\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \log(c_t^i) \tag{46}$$

subject to

$$\sum_{t=1}^{\infty} p_t[(1 + r_t)a_t^i + \varepsilon^i w_t - c_t^i - a_{t+1}^i] = 0 \tag{47}$$

where a_t^i is the asset holding of household of type i in period t and $\varepsilon^i \geq 0$ refers to labor productivity. We normalize so that average labor productivity equals one, that is $\sum q^i \varepsilon^i = 1$.

The first-order conditions are:

$$\frac{\beta^{t-1}}{c_t^i} - \lambda p_t = 0 \tag{48}$$

$$-p_{t+1}(1 + r_{t+1}) + p_t = 0 \tag{49}$$

where λ is the Lagrange multiplier of the budget constraint.

The problem of a representative firm is:

$$\max_{k_t, n_t} \{f(k_t, n_t) + (1 - \delta)k_t - w_t n_t - (1 + r_t)k_t\} \tag{50}$$

where k_t is the aggregate capital stock determined as $k_t = \sum q^i a_t^i$. Again, we get

$$r_t = f_{k_t} - \delta \tag{51}$$

$$w_t = f_{n_t} \tag{52}$$

The aggregate resource constraint is essentially the same as before:

$$c_t + k_{t+1} = f(k_t, 1) + (1 - \delta)k_t \tag{53}$$

4.2 Competitive equilibrium (with period 1 trading)

For given $\{a_1^i\}_{i=1}^{I}$, the competitive equilibrium now consists of an allocation $\{c_t^i, a_{t+1}^i\}_{t=1}^{\infty}$ for all $i = 1, 2, ..., I$, capital stocks $\{k_{t+1}\}_{t=1}^{\infty}$ and a price system $\{p_t, w_t, r_t\}_{t=1}^{\infty}$, such that for all i, $\{c_t^i, a_{t+1}^i\}_{t=1}^{\infty}$

solve the problem of a household of type i, prices are determined by (49), (51) and (52), the capital and labor markets clear and the aggregate resource constraint holds.

4.3 Representative household

We want to show that this model economy allows for a representative household. That is, aggregate variables can be found by considering the problem of a single representative household.

Consider first the steady state. The steady state version of the Euler equation is the same for all households and it reads as:

$$1 = \beta(1 + r) \tag{54}$$

Clearly, this equation pins down the steady state aggregate capital stock, denote it by k^*, (which depends on β alone) and hence also the interest rate and the wage rate (there is no point in determining p in steady state). Thus, the distribution doesn't matter for the aggregate steady state variables.

Consider then transitionary dynamics. The present value budget constraint can be written as

$$\sum_{t=1}^{\infty} p_t c_t^i - \sum_{t=1}^{\infty} \left[p_{t+1}(1 + r_{t+1}) - p_t\right] a_{t+1}^i = p_1 \left(1 + r_1\right) a_1^i + \sum_{t=1}^{\infty} p_t \varepsilon^i w_t \tag{55}$$

Using (49) this simplifies to

$$\sum_{t=1}^{\infty} p_t c_t^i = p_1 \left(1 + r_1\right) a_1^i + \sum_{t=1}^{\infty} p_t \varepsilon^i w_t. \tag{56}$$

Moreover, combining (48) and (49), we can show that

$$c_t^i = c_1^i \beta^{t-1} \prod_{s=2}^{t} \left(1 + r_t\right). \tag{57}$$

By plugging (57) into (56) it becomes clear that c_1^i is a linear function of a_1^i and ε^i. Then, by (57), c_t^i is also a linear function of a_1^i and ε^i. It follows that the the equilibrium consumption path is the one chosen by a representative household with initial asset holdings and labor productivity equal to the averages of initial asset holdings and labor productivities across all household types.

4.4 Indeterminate wealth distribution

The model does not determine the relative steady state wealth distribution. That is, any relative distribution of a^i can be a steady state distribution.

To see this, notice that if the interest rate solves (54), then $c_t^i = c_{t+1}^i$, for all households and steady state consumption level can be solved from the steady state version of the household budget constraint: it is the only consumption level that the household can keep constant without violating the budget constraint. In other words, it is the consumption level that is consistent with $a_t^i = a_{t+1}^i$. It follows that the asset distribution remains constant starting from any asset distribution $\{a^i\}_{i=1}^I$ such that $\sum q^i a^i = k^*$ where k^* is the unique steady state capital stock. (We must assume, however, that every household can afford positive consumption, that is, a^i cannot be arbitrarily small.)